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On the motion of rigid bodies in incompressible inviscid fluids of inhomogeneous density

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The motion of a rigid body in an inviscid incompressible fluid of inhomogeneous density is considered. The size of the body is taken small with respect to the length scale of the density variations; its shape is otherwise arbitrary. The force and the torque acting on the body in an arbitrary motion are derived from Hamilton's principle of least action, thus offering a variational derivation of Kirchhoff's equations, supplemented by the terms due to the density gradient. The force and the torque due to a density gradient are proportional to the gradient and quadratic in the velocity and the angular velocity. The same coefficients are shown to govern both the inertial behaviour of the body, i.e. the response to accelerations, and the effects of density gradients. The free motion of spheres and two-dimensional circular cylinders is shown to obey a condition akin to the Fermat principle in optics.

1. Introduction

A rigid body moving in an inviscid and incompressible fluid offers unique dynamic behaviour due to the reaction of the fluid on the body. This reaction amounts to increasing the inertia of the body, bringing an added mass as well as an added moment of inertia and an inertial coupling between translation and rotation. Unlike the mass of the solid, which is a scalar, the added mass is a tensor. Furthermore, the added inertia is proportional to the fluid density; therefore, it changes as the body moves through a fluid of non-uniform density. A thorough analysis of the distortion of the flow around the body due to the non-uniformity of the fluid has been given by Eames & Hunt (1997), who derived the lift and the drag force an axisymmetric body experiences when moving in a weak density gradient. They found very simple relations between the added mass coefficient and the lift and drag coefficients. Stimulated by this finding, we present here a derivation of the force and the torque a body of arbitrary shape experiences when moving and rotating in a weak density gradient. This derivation, based on the Hamiltonian mechanics of the combined system body plus fluid, uncovers the relation between the coefficients governing the inertia and those governing the force and the torque due to density gradients.

2. Motion equations for translation and rotation

The density gradient $\nabla\rho$ is considered weak when the variation of the fluid density ρ along the body is small, i.e. if Eames & Hunt's (1997) parameter $\varepsilon = a\rho^{-1}\nabla\rho$, a being the dimension of the body, is small: $\varepsilon \ll 1$. This condition is assumed to hold throughout this paper. At vanishing order in ε, the flow of the liquid due to the motion of the body can be shown to be potential and linear in both the translation and rotation velocity of the body at the considered time (Lamb 1932). Then the kinetic energy $T = \frac{1}{2}\int \rho \boldsymbol{v}^2 \mathrm{d}V$ of the fluid can be written, in Batchelor's (1967) notation,

$$T(\boldsymbol{X}, \boldsymbol{U}, \boldsymbol{R}, \boldsymbol{\Omega}) = \tfrac{1}{2}\rho V(\alpha_{ij}U_iU_j + \beta_{ij}U_i\Omega_j + \gamma_{ij}\Omega_i\Omega_j) \tag{1}$$

with $\rho = \rho(\boldsymbol{X})$, $\beta_{ij} = R_{ik}(t)R_{jl}(t)\beta^0_{kl}$ and correspondingly for α_{ij} and γ_{ij}, where V is the volume of the body, $\boldsymbol{X}$ is the position of a point of the body chosen as the origin of a frame attached to it, hereafter called the rotating frame, and $\boldsymbol{R}(t)$ is the time-dependent rotation tensor transforming the orientation of the rotating axes into that of the fixed axes. The tensors $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ are characteristics of the body; they rotate along with the rotating frame, so they assume the respective constant values $\boldsymbol{\alpha}^0$, $\boldsymbol{\beta}^0$ and $\boldsymbol{\gamma}^0$ in the rotating frame. These tensors depend on the shape of the body and are proportional to its length a at the power 0, 1 and 2 respectively. The kinetic energy depends on $\boldsymbol{X}$ through the variation of the local density ρ, i.e. at the considered precision, the kinetic energy is the same as if the fluid density around the body were uniform and equal to $\rho(\boldsymbol{X})$. Considering the distortion of the flow field due to the density gradient would lead to correcting expression (1) by terms smaller by a factor of order ε. Furthermore, persistent vorticity would make the kinetic energy (1) depend on the history of motion, instead of being a function of the instantaneous values of the dynamic variables. In the following, we will ignore such corrections, and concentrate on the analytical mechanics of systems obeying equation (1) at leading order in ε. The dynamical variables are the linear velocity $\boldsymbol{U}$ and the angular velocity $\boldsymbol{\Omega}$

$$\boldsymbol{U} = \dot{\boldsymbol{X}}, \quad \Omega_i = \tfrac{1}{2}\varepsilon_{ijk}R_{jl}\dot{R}_{kl}. \tag{2}$$

Using the angular velocity $\boldsymbol{\Omega}$ instead of the time derivative $\dot{\boldsymbol{R}}$ permits the tensorial complexity of our equations to be lowered. The three components of $\boldsymbol{\Omega}$ contain the same information as the three independent components of $\dot{R}_{ij} = \varepsilon_{ikl}\Omega_k R_{lj}$. The tensors $\boldsymbol{\alpha}$ and $\boldsymbol{\gamma}$ are symmetric and positive definite, and $\boldsymbol{\beta}$ has no general symmetry but it is such that T is a positive definite quadratic form $\boldsymbol{U}$ and $\boldsymbol{\Omega}$. Note that expression (1) is somewhat more general than the kinetic energy of a rigid body in vacuum, which would have an isotropic mass and an antisymmetric coupling tensor $\boldsymbol{\beta}$, reducing to zero if $\boldsymbol{X}$ is chosen as the centre of mass. In our case, the origin $\boldsymbol{X}$ can be chosen so as to cancel the antisymmetric part of $\boldsymbol{\beta}$, whereas the symmetric part does not depend on $\boldsymbol{X}$. The symmetric part of $\boldsymbol{\beta}$ couples the translation and the rotation, a characteristic of bodies possessing helical symmetry, like corkscrews and propellers (Lamb 1932). In the following, no special choice will be made for the origin of the rotating frame, so $\boldsymbol{\beta}$ is a general tensor.

The impulse $\boldsymbol{P}$ and the angular impulse $\boldsymbol{Q}$ are (Batchelor 1967)

$$P_i = \frac{\partial T}{\partial U_i} = \rho V(\alpha_{ij}U_j + \tfrac{1}{2}\beta_{ij}\Omega_j), \quad Q_i = \frac{\partial T}{\partial \Omega_i} = \rho V(\tfrac{1}{2}\beta_{ji}U_j + \gamma_{ij}\Omega_j). \tag{3}$$

These impulses are the quantities conjugate to the displacement and the rotation of the body in the Hamiltonian sense. In finite mechanical systems, these quantities coincide with the momentum $\int \rho \boldsymbol{v}\mathrm{d}V$ and the angular momentum $\int \rho((\boldsymbol{x}-\boldsymbol{X}) \times \boldsymbol{v})\mathrm{d}V$ respectively, $\boldsymbol{x}$ and $\boldsymbol{v}$ being the position vector and the velocity of mass element $\rho\mathrm{d}V$. The distinction matters in our system, since the momentum and the angular momentum lead to integrals of conditional convergence because of the behaviour of the fluid velocity at infinity (Lamb 1932). A fundamental point is that, after having established equation (1), it is legitimate of formulate a Hamiltonian description of our system based on the sole motion of the body. The infinitely many degrees of freedom of the fluid are then ignored, even though they are subjected to the non-holonomic constraint of incompressibility div $\boldsymbol{v} = 0$, which is a non-integrable relation between the velocities of the fluid particles. The validity of Hamiltonian mechanics for rigid bodies moving in fluids of homogeneous density can be directly proved from the equations of motion of the fluid (Lamb 1932). However, since we consider fluids of non-uniform density, we must resort to more general arguments using two particularities of our system. First, the constraints do not work: the rate of work of compression p div$\boldsymbol{v}$ vanishes, so the work the body performs on the fluid is entirely converted into kinetic energy, and the system is conservative. Second, the variables $\boldsymbol{X}$, $\boldsymbol{U}$, $\boldsymbol{R}$ and $\boldsymbol{\Omega}$ entering equation (1) are independent, so any variation of the real motion is a possible motion. As a consequence, since the constraint of incompressibility does not restrict the motion of the body, the associated forces of constraint vanish (see § 2.4 of Goldstein 1980). The dynamics of our system is thus identical with that of a holonomic system obeying equation (1).

The motion equation for the position variable is then the Lagrange equation

$$\boldsymbol{F} = \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial T}{\partial \boldsymbol{U}} - \frac{\partial T}{\partial \boldsymbol{X}} = \dot{\boldsymbol{P}} - \frac{\partial T}{\partial \boldsymbol{X}}, \tag{4}$$

where $\boldsymbol{F}$ is the force the body exerts on the fluid. The conservation of the impulse, which is the integral of the motion associated with the homogeneity of space (Landau & Lifshitz 1976†), no longer holds here. In other words, the body does not move in homogeneous empty space because it is 'clothed' by the surrounding fluid, which reacts on the body and contributes to its inertia. This particular space is not homogeneous but, since T is proportional to ρ at fixed $\boldsymbol{U}$ and $\boldsymbol{\Omega}$, one has

$$\frac{\partial T}{\partial \boldsymbol{X}} = T\nabla \ln \rho \quad \text{and} \quad \dot{\boldsymbol{P}} = \boldsymbol{F} + T\nabla \ln \rho. \tag{5}$$

The density gradient thus adds a term $T\nabla \ln \rho$ to the force. However, if the density depends on one Cartesian coordinate only, say $\rho = \rho(x_1)$, then the homogeneity of space in the 2- and 3-directions still holds and the components P_2 and P_3 are conserved whenever F_2 and F_3 vanish. The time derivative of the density gradient following the motion of the body, $\mathrm{d}\rho/\mathrm{d}t = \boldsymbol{U}\cdot\nabla\rho$, also contributes to the time derivative of the impulse, which can be written

$$\begin{aligned}\dot{\boldsymbol{P}}_i &= \rho V \frac{\mathrm{d}}{\mathrm{d}t}\left(\alpha_{ij}U_j + \tfrac{1}{2}\beta_{ij}\Omega_j\right) + P_j U_k \nabla_k \ln \rho \\ &= \rho V \left(\alpha_{ij}\frac{\Delta U_j}{\Delta t} + \tfrac{1}{2}\beta_{ij}\frac{\Delta \Omega_j}{\Delta t}\right) + \varepsilon_{ijk}\Omega_j P_k + P_i U_k \nabla_k \ln \rho,\end{aligned} \tag{6}$$

where $\Delta/\Delta t$ is the derivative with respect to the moving axes, such that $\Delta\boldsymbol{\alpha}/\Delta t$ $\Delta\boldsymbol{\beta}/\Delta t$ and $\Delta\boldsymbol{\gamma}/\Delta t$ vanish, and the term $\varepsilon_{ijk}\Omega_j P_k$ compensates for the rotation of the moving axes with respect to the fixed ones. The Lagrange equation for the position variables can now be written in full:

$$\begin{aligned}F_i &= \rho V \frac{\mathrm{d}}{\mathrm{d}t}\left(\alpha_{ij}U_j + \tfrac{1}{2}\beta_{ij}\Omega_j\right) + (P_i U_j - T\delta_{ij})\nabla_j \ln \rho \\ &= \rho V \left(\alpha_{ij}\frac{\Delta U_j}{\Delta t} + \tfrac{1}{2}\beta_{ij}\frac{\Delta \Omega_j}{\Delta t}\right) + \varepsilon_{ijk}\Omega_j P_k + U_j \nabla_j \rho V \left(\alpha_{ik}U_k + \tfrac{1}{2}\beta_{ik}\Omega_k\right)\end{aligned}$$

† Note that their definition of the force corresponds to our $\boldsymbol{F} + \partial T/\partial \boldsymbol{X}$.

$$-\tfrac{1}{2}\nabla_i\rho V(\alpha_{jk}U_jU_k + \beta_{jk}U_j\Omega_k + \gamma_{jk}\Omega_j\Omega_k). \tag{7}$$

The Lagrange equation for the angular variables should be written in terms of $\boldsymbol{R}$ and $\dot{\boldsymbol{R}}$, taking into account the fact that $\boldsymbol{R}$ and $\dot{\boldsymbol{R}}$ only have three independent components. Instead of this complicated procedure, we directly derive the angular motion equations from Hamilton's principle by considering the variation of the action integral in a motion varied through an infinitesimal time-dependent rotation $\delta\boldsymbol{R}(t)$, of angle $\delta\theta_i(t) = \frac{1}{2}\varepsilon_{ijk}R_{jl}\delta R_{kl}$, applied to the unvaried orientation $\boldsymbol{R}(t)$. The angular velocity at time t is then varied by

$$\delta\boldsymbol{\Omega} = \delta\dot{\boldsymbol{\theta}} + \delta\boldsymbol{\theta}\times\boldsymbol{\Omega}, \tag{8}$$

i.e. the variation $\delta\theta(t)$ both adds its time derivative to $\boldsymbol{\Omega}$ and rotates it. This can be seen by considering a vector $\boldsymbol{a}$ linked to the body, i.e. constant in the rotating frame and consequently such that $\dot{\boldsymbol{a}} = \boldsymbol{\Omega}\times\boldsymbol{a}$. The considered variation makes $\boldsymbol{a}$ vary by $\delta\boldsymbol{a} = \delta\boldsymbol{\theta}\times\boldsymbol{a}$, and the time derivative of the varied vector $\boldsymbol{a}+\delta\boldsymbol{a}$ can be written $\mathrm{d}(\boldsymbol{a}+\delta\boldsymbol{a})/\mathrm{d}t = \boldsymbol{\Omega}\times\boldsymbol{a}+\delta\dot{\boldsymbol{\theta}}\times\boldsymbol{a}+\delta\boldsymbol{\theta}\times(\boldsymbol{\Omega}\times\boldsymbol{a}) = (\boldsymbol{\Omega}+\delta\dot{\boldsymbol{\theta}}+\delta\boldsymbol{\theta}\times\boldsymbol{\Omega})\times(\boldsymbol{a}+\delta\boldsymbol{a})+O(\delta\boldsymbol{\theta}^2)$. The first term in parentheses of the last member is the varied angular velocity $\boldsymbol{\Omega}+\delta\boldsymbol{\Omega}$, hence equation (8), neglecting the second-order term $-(\delta\dot{\boldsymbol{\theta}}+\delta\boldsymbol{\theta}\times\boldsymbol{\Omega})\times\delta\boldsymbol{a} = O(\delta\boldsymbol{\theta}^2)$. The kinetic energy is thus varied though both the change of angular velocity $\delta\boldsymbol{\Omega}$ and through the change of orientation of the tensors $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$ and of the vector $\boldsymbol{\Omega}$. Since rotating $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$ and $\boldsymbol{\Omega}$, by $\delta\boldsymbol{\theta}$ while keeping $\boldsymbol{U}$ constant is equivalent to rotating $\boldsymbol{U}$ by $-\delta\boldsymbol{\theta}$ at constant $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$, $\boldsymbol{\gamma}$ and $\boldsymbol{\Omega}$, the first-order variation of T can be written

$$\begin{aligned}\delta T &= \frac{\partial T}{\partial\boldsymbol{\Omega}}\cdot\delta\dot{\boldsymbol{\theta}} - \frac{\partial T}{\partial\boldsymbol{U}}\cdot(\delta\boldsymbol{\theta}\times\boldsymbol{U}) = \boldsymbol{Q}\cdot\delta\dot{\boldsymbol{\theta}} - \boldsymbol{P}\cdot(\delta\boldsymbol{\theta}\times\boldsymbol{U})\\ &= \frac{\mathrm{d}}{\mathrm{d}t}(\boldsymbol{Q}\cdot\delta\boldsymbol{\theta}) - (\dot{\boldsymbol{Q}}+\boldsymbol{U}\times\boldsymbol{P})\cdot\delta\boldsymbol{\theta}.\end{aligned} \tag{9}$$

The potential energy W varies by

$$\delta W = -\boldsymbol{\Gamma}\cdot\delta\boldsymbol{\theta} \tag{10}$$

and Hamilton's principle (Goldstein 1980) states that the action integral S taken between any given times t_1 and t_2 is stationary with respect to any variation of the motion, so the first-order variation of S cancels. In the considered case of variation $\delta\boldsymbol{\theta}(t)$ of the orientation of the body, the first-order variation is

$$\begin{aligned}0 = \delta S &= \delta\int_{t_1}^{t_2}(T-W)\mathrm{d}t\\ &= [\boldsymbol{Q}\cdot\delta\boldsymbol{\theta}]_{t_1}^{t_2} + \int_{t_1}^{t_2}(-\dot{\boldsymbol{Q}} - \boldsymbol{U}\times\boldsymbol{P}+\boldsymbol{\Gamma})\cdot\delta\boldsymbol{\theta}\mathrm{d}t.\end{aligned} \tag{11}$$

Since $\delta\boldsymbol{\theta}(t)$ is arbitrary, a necessary condition for the stationarity of the action is that the term in parentheses vanishes, giving the angular motion equation

$$\boldsymbol{\Gamma} = \dot{\boldsymbol{Q}} + \boldsymbol{U}\times\boldsymbol{P}. \tag{12}$$

The time derivative $\dot{\boldsymbol{Q}}$ can be split into two components, in the same way as $\dot{\boldsymbol{P}}$ in equation (6),

$$\dot{\boldsymbol{Q}}_i = \rho V\left(\tfrac{1}{2}\beta_{ji}\frac{\Delta U_j}{\Delta t} + \gamma_{ij}\frac{\Delta\Omega_j}{\Delta t}\right) + \varepsilon_{ijk}\Omega_jQ_k + Q_iU_k\nabla_k\ln\rho, \tag{13}$$

giving the angular motion equation

$$\boldsymbol{\Gamma}_i = \rho V \left(\tfrac{1}{2}\beta_{ji}\frac{\Delta U_j}{\Delta t} + \gamma_{ij}\frac{\Delta \Omega_j}{\Delta t} \right) + \varepsilon_{ijk}\Omega_j Q_k + \varepsilon_{ijk} U_j P_k + Q_i U_k \nabla_k \ln\rho. \tag{14}$$

Equations (7) and (14) are the Kirchhoff equations for the motion of a solid moving in a perfect fluid (Lamb 1932), supplemented by the effect of the density gradient.

The conservation of energy holds as a consequence of the homogeneity of time (Landau & Lifshitz 1976), which is not broken by the existence of time-independent density gradients. Indeed, the rate of work of the force, obtained from equation (7), is

$$\boldsymbol{F}\cdot\boldsymbol{U} = \rho V \left(\alpha_{ij} U_i \frac{\Delta U_j}{\Delta t} + \tfrac{1}{2}\beta_{ij} U_i \frac{\Delta \Omega_j}{\Delta t} \right) + \varepsilon_{ijk} U_i \Omega_j P_k + (\boldsymbol{P}\cdot\boldsymbol{U} - \boldsymbol{T})\boldsymbol{U}\cdot\nabla \ln\rho \tag{15}$$

and the rate of work of the torque is, from equation (14),

$$\boldsymbol{\Gamma}\cdot\boldsymbol{\Omega} = \rho V \left(\tfrac{1}{2}\beta_{ji}\Omega_i \frac{\Delta U_j}{\Delta t} + \gamma_{ij}\Omega_i \frac{\Delta \Omega_j}{\Delta t} \right) + \varepsilon_{ijk}\Omega_i U_j P_k + \boldsymbol{Q}\cdot\boldsymbol{\Omega}\,\boldsymbol{U}\cdot\nabla \ln\rho. \tag{16}$$

Recalling that $\Delta\alpha_{ij}/\Delta t$, $\Delta\beta_{ij}/\Delta t$, and $\Delta\gamma_{ij}/\Delta t$ vanish, the total rate of work can be written

$$\begin{aligned} \boldsymbol{F}\cdot\boldsymbol{U} + \boldsymbol{\Gamma}\cdot\boldsymbol{\Omega} &= \tfrac{1}{2}\rho V \frac{\Delta}{\Delta t}(\alpha_{ij}U_iU_j + \beta_{ij}U_i\Omega_j + \gamma_{ij}\Omega_i\Omega_j) + (\boldsymbol{P}\cdot\boldsymbol{U} + \boldsymbol{Q}\cdot\boldsymbol{\Omega} - T)\,\boldsymbol{U}\cdot\nabla\ln\rho \\ &= \rho\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{T}{\rho}\right) + \frac{\boldsymbol{P}\cdot\boldsymbol{U} + \boldsymbol{Q}\cdot\boldsymbol{\Omega} - T}{\rho}\frac{\mathrm{d}\rho}{\mathrm{d}t}, \end{aligned} \tag{17}$$

where $\Delta/\Delta t$ before the first term in parentheses is substituted for the convected derivative in the fixed frame $\mathrm{d}/\mathrm{d}t$ because it acts on the scalar T/ρ. Using Euler's theorem on homogeneous functions applied to T as a quadratic function of $\boldsymbol{U}$ and $\boldsymbol{\Omega}$,

$$2T = \boldsymbol{P}\cdot\boldsymbol{U} + \boldsymbol{Q}\cdot\boldsymbol{\Omega}, \tag{18}$$

eventually yields the total rate of work as the time derivative of the kinetic energy

$$\boldsymbol{F}\cdot\boldsymbol{U} + \boldsymbol{\Gamma}\cdot\boldsymbol{\Omega} = \frac{\mathrm{d}T}{\mathrm{d}t}. \tag{19}$$

If the force $\boldsymbol{F}$ and the torque $\boldsymbol{\Gamma}$ vanish, then the kinetic energy is conserved, $T = \text{const.}$, and if they derive from a time-independent potential energy W, then the total energy $E = T + W$ is conserved: $\mathrm{d}E/\mathrm{d}t = 0$. A more general case is that of a body possessing its own inertia. Its kinetic energy $T_{body} = \frac{1}{2}M\boldsymbol{U}^2 + \frac{1}{2}I_{ij}\boldsymbol{\Omega}_i\boldsymbol{\Omega}_j$ must be added to T given by equation (1) without alteration of the form of this equation (the origin of the rotating axes is chosen as the centre of mass of the body, M being the mass and I_{ij} the tensor of inertia). $\boldsymbol{F}$ and $\boldsymbol{\Gamma}$ then become the force and the torque acting on the whole system (fluid + body). Then, if $\boldsymbol{F}$ and $\boldsymbol{\Gamma}$ derive from a time-independent potential W, the total energy $T + T_{body} + W$ is conserved. By the same token, equation (14) becomes Euler's equation for the motion of a rigid body of inertia tensor I_{ij} letting $\rho \to 0$ in equation (1). The last term of equation (14) then vanishes, and so does the last but one term, due to the fact that the impulse (3) becomes $\boldsymbol{P}_{body} = M\boldsymbol{U}$, collinear with the velocity $\boldsymbol{U}$.

3. Forces and torques in inhomogeneous fluids

We now focus on the specific effect of the density gradient. The force $\boldsymbol{\Phi}$ and the torque $\boldsymbol{\Xi}$ the fluid exerts on the body due to the density gradient are, from equations

(7), (14) and (3),

$$\left.\begin{aligned}\Phi_i &= -U_j\nabla_j\rho V(\alpha_{ik}U_k + \tfrac{1}{2}\beta_{ik}\Omega_k) + \tfrac{1}{2}\nabla_i\rho V(\alpha_{jk}U_jU_k + \beta_{jk}U_j\Omega_k + \gamma_{jk}\Omega_j\Omega_k),\\ \Xi_i &= -U_j\nabla_j\rho V(\tfrac{1}{2}\beta_{ki}U_k + \gamma_{ik}\Omega_k).\end{aligned}\right\} \tag{20}$$

$\boldsymbol{\Phi}$ and $\boldsymbol{\Xi}$ are quadratic function of $\boldsymbol{U}$ and $\boldsymbol{\Omega}$, linear in $\nabla\rho$, and do not depend on the accelerations $\dot{\boldsymbol{U}}$ and $\dot{\boldsymbol{\Omega}}$. Note that $\boldsymbol{\Phi}$ and $\boldsymbol{\Xi}$, being construed as reactions of the fluid on the body, are defined with a sign opposite of that $\boldsymbol{F}$ and $\boldsymbol{\Gamma}$.

Let us first consider the case of motion without rotation. Taking the 1-axis of the fixed frame along the density gradient evaluated at the considered point and the 2-axis in the plane $(\boldsymbol{U}, \nabla\rho)$, one has

$$\nabla\rho = \nabla\rho\begin{pmatrix}1\\0\\0\end{pmatrix},\quad \boldsymbol{U} = \begin{pmatrix}U_1\\U_2\\0\end{pmatrix},\quad \boldsymbol{\Omega} = 0. \tag{21}$$

Equation (20) then gives

$$\boldsymbol{\Phi} = -V\nabla\rho\begin{pmatrix}\frac{1}{2}\alpha_{11}U_1^2 - \frac{1}{2}\alpha_{22}U_2^2\\ \alpha_{12}U_1^2 + \alpha_{22}U_1U_2\\ \alpha_{13}U_1^2 + \alpha_{23}U_1U_2\end{pmatrix},\quad \boldsymbol{\Xi} = -\tfrac{1}{2}V\nabla\rho U_1\begin{pmatrix}\beta_{11}U_1 + \beta_{21}U_2\\ \beta_{12}U_1 + \beta_{22}U_2\\ \beta_{13}U_1 + \beta_{23}U_2\end{pmatrix}. \tag{22}$$

In general the orientation of $\nabla\rho$ varies in space, so the frame within which the two preceding equations are written is locally defined.

If the velocity is parallel to the density gradient, the force and the torque are

$$\boldsymbol{U} = \begin{pmatrix}U\\0\\0\end{pmatrix} \Rightarrow \boldsymbol{\Phi} = -V\nabla\rho U^2\begin{pmatrix}\frac{1}{2}\alpha_{11}\\ \alpha_{12}\\ \alpha_{13}\end{pmatrix},\quad \boldsymbol{\Xi} = -\tfrac{1}{2}V\nabla\rho U^2\begin{pmatrix}\beta_{11}\\ \beta_{12}\\ \beta_{13}\end{pmatrix}. \tag{23}$$

If the cross-terms vanish, $\alpha_{12} = \alpha_{13} = 0$, one has simply $\boldsymbol{\Phi} = -\frac{1}{2}V\nabla\rho\alpha_{11}U^2$, so the body experiences a force directed towards the lighter fluid when moving parallel to the density gradient, whether it moves towards the denser or towards the lighter fluid. The component Φ_1 has sign opposite that of F_1, α_{11} being positive. This is understandable in terms of energy conservation: the reaction of the fluid on the body counteracts the variation of kinetic energy $\frac{1}{2}\rho V\alpha_{11}U^2$ due to the change in ρ by increasing (decreasing) the velocity when the body moves towards lighter (denser) regions. A body possessing helical symmetry about the 1-axis, such as a corkscrew, is such that $\beta_{12} = \beta_{13} = 0$ and $\beta_{11} \neq 0$ (the usual right-handed corkscrew has $\beta_{11} < 0$, so the kinetic energy T is lowered by a rotation $\Omega_1 = -\boldsymbol{U}_1\alpha_{11}/\beta_{11}$). Such an object experiences a torque $\boldsymbol{\Xi} = -\frac{1}{2}V\nabla\rho\beta_{11}U^2$, so a right-handed corkscrew experiences a positive torque when moving at constant velocity towards denser fluid, this torque tending to set the body in a rotation that lowers T, as expected from the conservation of energy.

A velocity perpendicular to the density gradient generates a force and a torque

$$\boldsymbol{U} = \begin{pmatrix}0\\U\\0\end{pmatrix} \Rightarrow \boldsymbol{\Phi} = \tfrac{1}{2}V\alpha_{22}\nabla\rho U^2,\quad \boldsymbol{\Xi} = 0. \tag{24}$$

This holds true for bodies of arbitrary shape. The body thus experiences no torque, and a lift force normal to the velocity, parallel to the density gradient and directed towards the denser fluid.

4. Trajectories of spheres and two-dimensional circular cylinders

Our formalism readily applies to the free motion of a sphere or, in two dimensions, the case of a circular cylinder, taking $\boldsymbol{X}$ as the position of the centre, so that $\boldsymbol{\alpha} = \alpha\boldsymbol{\delta}$, $\boldsymbol{\beta} = \boldsymbol{\gamma} = 0$. For the sphere, one has $\alpha = \frac{1}{2}$, and for the cylinder $\alpha = 1$. The angular coordinates are irrelevant and, if no external forces act on the system, the force $\boldsymbol{F}$ defined in equation (4) as the force the body exerts on the fluid is such that the total force vanishes: $\boldsymbol{F} + M\dot{\boldsymbol{U}} = 0$, M being the mass of the body. The trajectories then can be directly determined using Hamilton's principle together with the conservation of energy, without actually solving the dynamical problem. Introducing the position vector $\boldsymbol{X}(l)$ as a function of the curvilinear length l measured along the trajectory, the kinetic energy T of the total system (body and fluid) is

$$T = \tfrac{1}{2}\widetilde{m}\boldsymbol{U}^2 = \tfrac{1}{2}\widetilde{m}\left(\frac{\mathrm{d}l}{\mathrm{d}t}\right)^2 \Rightarrow \mathrm{d}t = \left(\frac{\widetilde{m}}{2T}\right)^{1/2}\mathrm{d}l \quad \text{where} \quad \widetilde{m} = \alpha\rho V + M; \tag{25}$$

$\widetilde{m}$ is the effective mass of the sphere or the cylinder, 'clothed' by the fluid. In the free motion we consider, the kinetic energy T is a constant, and the action integral S for the whole system, taken between times t_1 and t_2 can be written in terms of J, the curvilinear integral of $\sqrt{\widetilde{m}}$ performed along the trajectory:

$$S = T\int_{t_1}^{t_2}\mathrm{d}t = (\tfrac{1}{2}T)^{1/2}J, \quad \text{where} \quad J = \int_{X_1}^{X_2}\widetilde{m}^{1/2}\mathrm{d}l. \tag{26}$$

The stationarity condition for S implies a condition on J, i.e. a condition on the trajectory alone, irrespective of the velocity at which this trajectory is followed. Consider therefore a variation $\delta\boldsymbol{X}(t)$ of the trajectory, keeping the same the initial and final points: $\delta\boldsymbol{X}(t_1) = \delta\boldsymbol{X}(t_2) = 0$. From now on, we restrict our attention to varied trajectories followed at such a velocity as to keep T independent of time, but not necessarily equal to its value in the unvaried motion. In fact, the variation δT is such that the varied trajectory is still covered within the time span $t_2 - t_1$. The first-order variations of the three members of equation (26) are such that

$$\delta S = (t_1 - t_2)\delta T = J\delta\left((\tfrac{1}{2}T)^{1/2}\right) + (\tfrac{1}{2}T)^{1/2}\delta J; \tag{27}$$

thus the stationarity condition $\delta S = 0$ implies that both T and J are stationary:

$$\delta T = 0 \quad \text{and} \quad \delta J = 0. \tag{28}$$

The stationarity condition for the integral J is analogous to Fermat's principle stating the stationarity of the optical path in a medium of optical index $n = \widetilde{m}^{1/2}$. Introducing the unit tangent $\boldsymbol{t} = \mathrm{d}\boldsymbol{X}/\mathrm{d}l$, the stationarity condition for J can be written

$$\begin{aligned} 0 = \delta J &= \int_{X_1}^{X_2}\left(\delta\boldsymbol{X}\cdot\nabla\widetilde{m}^{1/2} + \widetilde{m}^{1/2}\boldsymbol{t}\cdot\frac{\mathrm{d}\delta\boldsymbol{X}}{\mathrm{d}l}\right)\mathrm{d}l \\ &= [\widetilde{m}^{1/2}\boldsymbol{t}\cdot\delta\boldsymbol{X}]_{X_1}^{X_2} + \int_{X_1}^{X_2}\left(\nabla\widetilde{m}^{1/2} - \frac{\mathrm{d}}{\mathrm{d}l}(\widetilde{m}^{1/2}\boldsymbol{t})\right)\cdot\delta\boldsymbol{X}\mathrm{d}l. \end{aligned} \tag{29}$$

The integrated term vanishes, and the condition that the integrand $\nabla\widetilde{m}^{1/2}$–$(\mathrm{d}/\mathrm{d}l)(\widetilde{m}^{1/2}\boldsymbol{t})$ must vanish identically gives the equation of the trajectory, which can be written using the radius of curvature R and the unit normal $\boldsymbol{n} = R\mathrm{d}\boldsymbol{t}/\mathrm{d}l$ (Landau & Lifshitz 1984):

$$\frac{\boldsymbol{n}}{R} = \nabla\ln\widetilde{m}^{1/2} - \boldsymbol{t}\,\boldsymbol{t}\cdot\nabla\ln\widetilde{m}^{1/2}, \quad R^{-1} = \boldsymbol{n}\cdot\nabla\ln\widetilde{m}^{1/2}. \tag{30}$$

The trajectory thus determined does not depend on the magnitude of the velocity, i.e. it is determined by one of its points and by the direction $\boldsymbol{t}$ of the velocity at that point. It must be noted that bodies of different diameters, with the same inner density M/V, have effective mass $\widetilde{m}(\boldsymbol{X}) = V(\rho(\boldsymbol{X}) + M/V)$ proportional to their respective volume, so the vector $\nabla \ln \widetilde{m}^{1/2}$ does not depend on V. The trajectory of a sphere or a circular cylinder thus only depends on its inner density.

The full dynamical problem requires solving equation (7) in the considered case of a sphere or a two-dimensional circular cylinder, where this equation reduces to

$$\dot{\boldsymbol{U}} = \tfrac{1}{2}\boldsymbol{U}^2 \nabla \ln \widetilde{m} - \boldsymbol{U}\ \boldsymbol{U} \cdot \nabla \ln \widetilde{m}. \tag{31}$$

Since the acceleration is a quadratic function of the velocity, the trajectory does not depend on the magnitude of the velocity, as already noted. Introducing the axes 1 and 2 defined in equation (21) yields

$$(\dot{\boldsymbol{U}})_1 = \tfrac{1}{2}\nabla \ln \widetilde{m}(U_2^2 - U_1^2), \quad (\dot{\boldsymbol{U}})_2 = -\nabla \ln \widetilde{m}\, U_1 U_2, \tag{32}$$

where $(\dot{\boldsymbol{U}})_1$, the 1-component of the acceleration, must be distinguished from $\dot{U}_1$, the time derivative of the 1-component of the velocity.

From now on, consider a density gradient which is uniform in direction, i.e. when $\rho = \rho(x_1)$ in a uniquely defined frame. Then $(\dot{\boldsymbol{U}})_1$ and $(\dot{\boldsymbol{U}})_2$ coincide with $\dot{\boldsymbol{U}}_1$ and $\dot{\boldsymbol{U}}_2$ respectively. Since the density gradient lies along the 1-axis, and according to equation (5), the 2-component of the impulse of the total system (body + fluid) is conserved:

$$P_2 = \widetilde{m} U_2 = \text{const.} \tag{33}$$

and the conservation of energy can be written

$$2T = \widetilde{m}(U_1^2 + U_2^2) = \frac{P_2^2}{\widetilde{m}} + \widetilde{m} U_1^2 = \text{const.} \tag{34}$$

U_1^2 being positive and $P_2 = \widetilde{m} U_2$ constant, the motion is limited to the region of space where

$$\widetilde{m} \geqslant \frac{P_2^2}{2T}. \tag{35}$$

The trajectory of the body can have a tangent contact with the surface $\widetilde{m} = P_2^2/2T$, with $U_1 = 0$ and $U_2 = 2T/P_2$ at the contact point, and the body experiences a lift force given by equation (24), bending the trajectory towards the denser regions, in accordance with equation (30). This is an analogue of the mirage in optics, which follows from the Fermat principle similar to our equations (27) and (29). Equations (33) and (34) give the velocity, expressed as a function of $\widetilde{m}$, i.e. of $\boldsymbol{X}$

$$U_1 = \pm \left(\frac{2T}{\widetilde{m}} - \left(\frac{P_2}{\widetilde{m}} \right)^2 \right)^{1/2}, \quad U_2 = \frac{P_2}{\widetilde{m}}, \tag{36}$$

where the $\pm$ sign refers to the two branches of the trajectory, separated by the contact with the surface $\widetilde{m} = P_2^2/2T$. The time can be expressed as a function of X_1 by integrating the first equation (36), and then X_2 follows from the second equation (36):

$$t(X_1) = \pm \int_0^{X_1} \left(\frac{2T}{\widetilde{m}(\xi)} - \left(\frac{P_2}{\widetilde{m}(\xi)} \right)^2 \right)^{-1/2} \mathrm{d}\xi, \quad X_2 = P_2 \int_0^t \frac{\mathrm{d}t'}{\widetilde{m}(t')}, \tag{37}$$

where the contact with the surface $\widetilde{m} = P_2^2/2T$ defines the origin of time and space. Equations (36) and (37) permit the motion to be determined without actually integrating the motion equations (32).

A special case in which analytic integration is simple is that of an effective mass $\widetilde{m}(\boldsymbol{X})$ varying according to

$$\widetilde{m} = \frac{P_2^2}{2T} \mathrm{e}^{X_1/\lambda}, \quad \text{such that} \quad \nabla \ln \widetilde{m} = \lambda^{-1} = \text{const.} \tag{38}$$

Condition (35) then confines the motion to the half-space $x_1 \geqslant 0$, and equation (37) readily gives

$$t = \pm \frac{\lambda P_2}{T} (\mathrm{e}^{X_1/\lambda} - 1)^{1/2}, \quad X_1 = \lambda \ln \left(\left(\frac{Tt}{\lambda P_2} \right)^2 + 1 \right), \quad X_2 = 2\lambda \arctan \frac{Tt}{\lambda P_2}. \tag{39}$$

Eliminating the time from the last two equations yields the equation of the trajectory $X_1 = -2\lambda \ln \cos X_2/2\lambda$. This trajectory has a finite extent in the 2-direction, from $X_2 = -\pi\lambda$ to $\pi\lambda$.

Consider another case of integrable motion, the linear profile of effective mass, of uniform gradient μ:

$$\widetilde{m} = \mu(X_1 + \lambda), \quad \text{with} \quad \lambda = \frac{P_2^2}{2\mu T}. \tag{40}$$

The motion is again restricted to the half-space $x_1 \geqslant 0$ and, introducing the dimensionless time τ, one gets

$$\left.\begin{array}{l} \tau = \dfrac{3T^2\mu}{P_2^3} t = \frac{1}{2}\left(3 + \dfrac{X_1}{\lambda}\right)\left(\dfrac{X_1}{\lambda}\right)^{1/2}, \\ \left.\begin{array}{l} X_1 = \lambda\Theta^2, \\ X_2 = 2\lambda\Theta, \end{array}\right\} \quad \text{with} \quad \Theta = \left((1+\tau^2)^{1/2} + \tau\right)^{1/3} - \left((1+\tau^2)^{1/2} - \tau\right)^{1/3}. \end{array}\right\} \tag{41}$$

The trajectory is a parabola, described at non-uniform velocity. Unlike the preceding one, this trajectory has infinite extent in the 2-direction.

Our results must now be compared to those obtained by Eames & Hunt (1997). These authors introduce three coefficients: the added-mass coefficient C_M which governs the kinetic energy, and the lift and drag coefficients C_L and C_D giving the lift and drag forces respectively. C_M depends on the relative orientation of the body and the velocity, in our notation $C_M(\boldsymbol{U}) = \alpha_{ij} U_i U_j / U^2$. They derive the relations $C_L = \frac{1}{2} C_M (\boldsymbol{U} \perp \nabla\rho)$ and $C_D = \frac{1}{2} C_M (\boldsymbol{U} \parallel \nabla\rho)$, i.e. $C_L = \alpha_{22}$ and $C_D = \alpha_{11}$ in the orientation of equation (21), in agreement with our equations (24) and (23) respectively, in the case of bodies symmetric about the direction of $\boldsymbol{U}$, such that the off-diagonal components of $\boldsymbol{\alpha}$ vanish. For two-dimensional bodies possessing the same symmetry, they get $C_L = \frac{1}{2}(1 + C_M(\boldsymbol{U} \perp \nabla\rho))$ and $C_D = \frac{1}{2} C_M (\boldsymbol{U} \parallel \nabla\rho)$, which implies a discontinuity in the lift force when going from elongated three-dimensional to truly two-dimensional bodies. Our analysis however shows that the relations they obtained in the three-dimensional case necessarily hold for two-dimensional bodies as well.

In conclusion, our formalism gives a general and compact derivation, valid for bodies of arbitrary shape, of the force and the torque due to a weak density gradient, demonstrating that the same coefficients $\boldsymbol{\alpha}$, $\boldsymbol{\beta}$ and $\boldsymbol{\gamma}$ govern both the effects of the density gradient and the inertial reaction of the fluid on the body. In addition it permits us to formulate the fundamental mechanical laws of conservation valid in this particular system. Besides their fundamental interest, these laws allow in some cases a direct determination of the motion.

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